Uphills, Downhills and the Boston Marathon

by Bob Baumel

Some critics of TAC's new Drop/Separation rule have argued that although the Boston Marathon drops 3.3 meters per kilometer (which exceeds the 2 m/km limit of the old rule as well as the 1 m/km limit in the new rule), the uphills on this course, especially the famed "Heartbreak Hill," have such a devastating effect as to completely wipe out any aid provided by the drop. I will attempt here to evaluate this argument quantitatively. I conclude that, even under very conservative assumptions, Boston's downhills do aid performances considerably more than its uphills hurt them.

The mathematical framework for discussing this type of question was provided by my article in Jan '89 *Measurement News* entitled "Hill Effect to Second Order." I began that article with a race director's hypothetical claim: "Sure my course drops 500 meters, but it's really tough because it climbs 1000 meters before falling 1500 meters!" Unfortunately, it's likely that few people read all the way through that article, due to my use of integral calculus (although the math really wasn't as involved as in my subsequent "Physiological Model" article in Nov '89 MN).

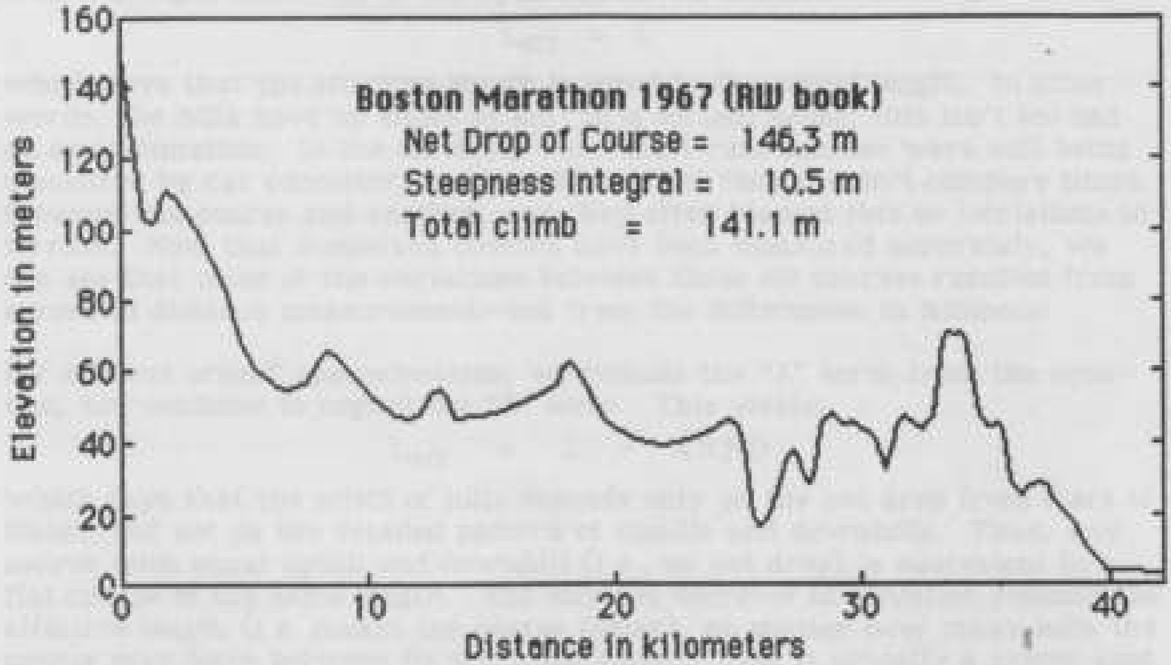
One person who obviously did thoroughly read and understand my Jan '89 article was Alan Jones, who wrote a follow-up article in July '89 MN with the same title: "Hill Effect to Second Order". Alan applied my equations to a local 20 km course, and found that the results agreed almost exactly with the actual difference between his race times on this hilly 20 km course and on a different (flat) 20 km course. This "experiment of one" doesn't really prove anything, but does lend credibility to the equations.

Unfortunately, neither Alan nor I applied the equations to any *major* courses such as Boston. (Applying the equations to any given course requires a very painstaking process of data collection from topographic maps.) I did urge in my Jan '89 article that the data be gathered for the Boston and St. George Marathons. And Alan announced in his July '89 article that he would try doing it for Boston. I assume that Alan just couldn't muster the energy to pore over the topo maps to obtain the 42 kilometers of data.

Just a few days ago, Pete Riegel reminded me that detailed topographic data for the Boston Marathon already exists in an old *Runner's World* booklet called *The Boston Marathon*. As it happens, I already owned a copy of this booklet, although I had completely forgotten about it. (Pete has clearly been around the running scene longer than I have, as he has the original 1972 edition of this booklet, while I have only the revised 1974 edition.)

The topographic data in this old *Runner's World* booklet was obtained by Rick Levy for the **1967** Boston course, which wasn't *exactly* the same as the present course. (That 1967 course dropped 3.5 m/km while the present course drops 3.3 m/km.) But it was probably close enough that we can use it to investigate the extent to which uphills cancel the downhills.

The following diagram was obtained by digitizing the profile chart on page 18 of that old *Runner's World* booklet, and feeding the result into the program I wrote at the time I prepared my Jan '89 *Measurement News* article:



The legend on this diagram includes the course's Net Drop and its calculated "Steepness Integral," which are quantities needed in the equation presented in my Jan '89 MN article. I will now review that equation, which can be written in the form:

$$L_{eff} = L - A \times ND + B \times SI \qquad (1)$$

where

- L is the course's *actual* length.
- L_{eff} is the course's "effective" length; i.e., the length of the perfectly flat course that would produce times identical to those run on the actual race course.
- ND is the course's Net Drop; i.e., the net decrease in elevation from Start to Finish (negative in case of net rise).
- SI is the quantity I call the "Steepness Integral," which measures the extent to which the course contains steep grades. (See Appendix for mathematical definition.)
- A and B are numerical coefficients whose values may be derived from exercise physiology exeriments involving oxygen uptake measurements on inclined treadmills. The actual method of deriving A and B from such data was explained in my Nov '89 MN article.

What does Equation (1) mean? The three terms on the right-hand-side of this equation can be thought of as "zeroth order", "first order", and "second order" terms respectively. Let me try to explain these three successive levels of approximation:

In the "zeroth order" approximation, we neglect both the "A" and "B" terms from the right-hand-side of the equation, so we are left with only:

$$L_{eff} = L$$

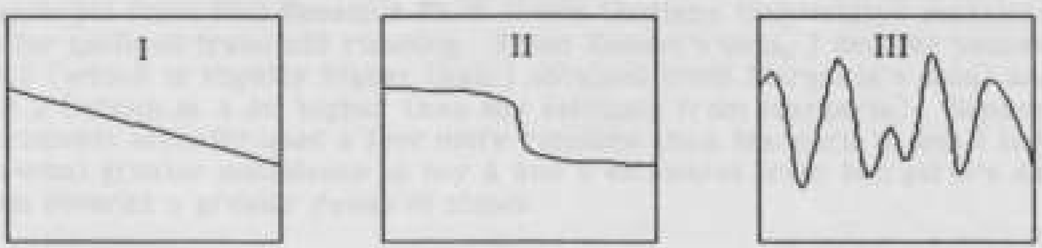
which says that the effective length is equal to the actual length. In other words, the hills have no effect at all! In a certain sense, this isn't too bad an approximation: In the old days when most race courses were still being measured by car odometer, runners knew that they couldn't compare times between one course and another, and they often blamed this on variations in terrain. Now that numerous courses have been measured accurately, we can see that *most* of the variations between those old courses resulted from errors in distance measurement—not from the differences in hilliness!

For a "first order" approximation, we include the "A" term from the equation, but continue to neglect the "B" term. This yields:

$$L_{eff} = L - A \times ND$$

which says that the effect of hills depends only on the net drop from start to finish, but not on the detailed pattern of uphills and downhills. Thus, any course with equal uphill and downhill (i.e., no net drop) is equivalent to a flat course of the same length. And any net decrease in elevation reduces the effective length (i.e. makes the course faster), no matter how many hills the course may have between its start and finish. This is actually a pretty good approximation if the course has only *gentle* slopes; in that case, the energy saved in descending one meter does almost completely cancel the extra energy used in climbing one meter.

But if the course has sufficiently steep grades, we need *all three* terms from equation (1). This is the "second order" approximation. Whereas the "A" term expresses the notion that equal uphills and downhills exactly cancel each other, the "B" term indicates the residual amount by which the uphills and downhills *don't* cancel each other. The Steepness Integral "SI" indicates the extent to which the course has steep grades. A large SI always increases the effective length; i.e., makes the course slower. The following diagrams illustrate three courses, all with the same net drop:



Course I descends uniformly and gently from start to finish, so its SI would be very small. The "first order" approximation would be very accurate for course I; i.e., the effect of the downhill can be calculated accurately knowing only the course's net drop.

The drop in course II is not uniform, but is concentrated in one steep descent somewhere in the middle. Because of this steep grade, course II has a bigger

Steepness Integral than course I, and would therefore produce slower times. But as course II is still a totally *downhill* course, it's surely a lot faster than a *flat* course of the same length. In all likelihood, the steepness effect ("B" term) for course II would only slightly reduce the aid calculated from the net drop ("A" term).

Course III has lots of steep uphills and downhills, so its SI would be much bigger than for courses I and II. In this case, the increased difficulty due to the steepness factor ("B" term) would eliminate a large portion of the aid calculated from the net drop ("A" term). In fact, it *could* happen that the "B" term completely overwhelms the "A" term, so that course III (in spite of its net drop) might actually be slower than a flat course of the same length.

Now that I've explained all the preliminaries, let's return to the Boston Marathon. From the profile diagram presented earlier, we already know its Net Drop and Steepness Integral (at least for the 1967 course); namely, ND = 146.3 m and SI = 10.5 m. All we need now are values for the coefficients A and B. In my Jan '89 MN article, I used A = 4.5 and B = 5. Substitution of these values in equation (1) yields (with all distances in meters):

$$\begin{aligned} L_{eff} &= 42195 - 658 + 52 \\ &= 42195 - 606 \\ &= 41589 \end{aligned}$$

Thus, the steepness term eliminates only about 8% of the aid predicted from the net drop, and the overall effect of the hills is equivalent to shortening the course 606 meters (for a time reduction of 1 min 50 sec at world-class speed of 5.5 m/s).

But perhaps I'm being too hard on Boston, as the values of A and B are really quite uncertain (especially the value of B). My values in the Jan '89 article were derived from the article: R. Margaria, P. Cerretelli, P. Aghemo, G. Sassi, "Energy cost of running," Journal of Applied Physiology, v. 18, 1963, p. 367. Actually, I noted that from Margaria's data, I had derived estimates of B ranging from 4 to 10, but I considered B = 5 as a "best" estimate.

While the above-mentioned article of Margaria et. al. is probably the classic reference on energy cost of uphill and downhill running, it is still desirable to check other (independent) data on the subject. Recently, Jack Moran sent me excerpts from Phil Henson's Ph.D. thesis (Indiana University) containing data for inclined treadmill running. From Henson's data, I derived values: A = 5.6 (which is slightly higher than I obtained from Margaria's data) and B = 18.2 (which is *a lot* higher than my estimate from Margaria). Henson's experiments actually used a few more runners than Margaria's, but I have somewhat greater confidence in my A and B estimates from Margaria's data, which covered a greater range of slopes.

In any case, to be as generous as possible to supporters of Boston, suppose we re-calculate its effective length using the smallest possible value for A, and largest possible value for B. In particular, let's use A = 4, which is equivalent to Pete Riegel's estimate of the slope effect in Sept '89 MN (page 6), and B = 18.2 which I derived from Phil Henson's data. With these figures, we obtain:

$$
\begin{aligned}
L_{eff} &= 42195 - 585 + 191 \\
&= 42195 - 394 \\
&= 41801
\end{aligned}
$$

By this calculation, the steepness effect would eliminate about 33% of the aid predicted from net drop, and the overall hill effect is equivalent to shortening the course 394 meters (equivalent time reduction: 1 min 12 s). For reference, I note that an effective shortening of 394 m would be about 2.7 times as great as the shortness found in validating the 1981 New York Marathon.

In spite of all this analysis, some defenders of Boston might still try arguing that Heartbreak Hill has a far more devastating effect than indicated by its steepness, because of its particular location at a point where many marathoners "hit the wall." In response, I point out that a runner in top condition who is having a peak performance (the sort of performance that sets records) does *not* "hit the wall." In an optimally-paced performance, according to my Nov '89 MN article, the runner speeds up on the downhills, and slows down on the uphills, just enough to maintain constant energy output, and is not fully spent until the very end of the race.

Appendix: Definition of Steepness Integral

The "Steepness Integral" is defined as

$$SI = \int_0^L \left(\frac{dy}{dx}\right)^2 dx \qquad (A1)$$

where "x" (restricted to the interval $0 \le x \le L$) denotes distance along the course; and "y" is the elevation at position x. The *derivative* "dy/dx" is the course's local slope at position x. Since the integrand consists of the *squared* slope, SI is always *non-negative*. The biggest contributions to this integral come from the regions of steepest slope (either uphill or downhill). Hence, the name "Steepness Integral."

My legend on the Boston profile diagram includes the course's "Total Climb" in addition to its net drop and steepness integral. The Total Climb, denoted "TC", is a quantity popularized by Ken Young, and is found by adding up all the uphill elevation changes on the course. It can be shown rigorously that a course's SI, TC and ND always satisfy the inequality:

$$SI \ge \frac{(2 \times TC + ND)^2}{L} \qquad (A2)$$

In practice, the left side of (A2) is usually about 2.5 times as great as the right side, which suggests that we might try replacing equation (1) with:

$$L_{eff} \approx L - A \times ND + 2.5\,B\,\frac{(2 \times TC + ND)^2}{L} \qquad (A3)$$

which would be very handy because it requires a lot less detailed analysis of the topographic maps to figure a course's TC than to compute its SI.